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A Study on the Fractional Differential Problem of Some Type of Fractional Trigonometric Function

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Abstract: **In this paper, we obtain the formula of arbitrary order fractional derivative of some type of fractional trigonometric function by using the fractional Fourier series theory. Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions play important roles in this article. In fact, our result is a generalization of the result in classical calculus.**

Keywords: **Fractional derivative, fractional trigonometric function, fractional Fourier series, Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, viscoelasticity, physics, mechanics, biology, electrical engineering, control theory, and so on [1-9].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [10-14]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, we obtain arbitrary order fractional derivative of the following fractional trigonometric function:

$$[a+bcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)},$$

where $0<\alpha\leq 1$, a,b are real numbers, $a>b$, and $b\neq 0$. Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions play important roles in this article. In fact, our result is a generalization of ordinary calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([15])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ . \quad (1)$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer n, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the n-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([16])**:** *If α, β, x_0, C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

In the following, the definition of fractional analytic function is introduced.

Definition 2.3 ([17])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([18])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{7}$$

Definition 2.5 ([19])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha k}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k}. \tag{11}$$

Definition 2.6 ([20])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional cosine and sine function are defined as follows:

$$cos_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^{k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2k}, \tag{12}$$

and

$$sin_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^{k}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(2k+1)}. \tag{13}$$

Definition 2.7 ([21])**:** Let $0<\alpha\leq 1$, and $f_{\alpha}(x^{\alpha}), g_{\alpha}(x^{\alpha})$ be two α-fractional analytic functions. Then $\left(f_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}n}=f_{\alpha}(x^{\alpha})\otimes_{\alpha}\cdots\otimes_{\alpha}f_{\alpha}(x^{\alpha})$ is called the nth power of $f_{\alpha}(x^{\alpha})$. On the other hand, if $f_{\alpha}(x^{\alpha})\otimes_{\alpha}g_{\alpha}(x^{\alpha})=1$, then $g_{\alpha}(x^{\alpha})$ is called the $\otimes_{\alpha}$ reciprocal of $f_{\alpha}(x^{\alpha})$, and is denoted by $\left(f_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}(-1)}$.

Definition 2.8: If the complex number $z=p+iq$, where p,q are real numbers, and $i=\sqrt{-1}$. p, the real part of z, is denoted by $\text{Re}(z)$; q the imaginary part of z, is denoted by $\text{Im}(z)$.

Definition 2.9: The smallest positive real number T_{α} such that $E_{\alpha}(iT_{\alpha})=1$, is called the period of $E_{\alpha}(ix^{\alpha})$.

Proposition 2.10 (fractional Euler's formula)**:** *Let* $0<\alpha\leq 1$, *then*

$$E_{\alpha}(ix^{\alpha})=cos_{\alpha}(x^{\alpha})+isin_{\alpha}(x^{\alpha})\,. \tag{14}$$

Proposition 2.11 (fractional DeMoivre's formula)**:** *Let* $0<\alpha\leq 1$, *and* k *be a positive integer, then*

$$[cos_{\alpha}(x^{\alpha})+isin_{\alpha}(x^{\alpha})]^{k}=cos_{\alpha}(kx^{\alpha})+isin_{\alpha}(kx^{\alpha}). \tag{15}$$

III. MAIN RESULTS

In this section, we obtain any order fractional derivative of some type of fractional trigonometric function. At first, we find the fractional Fourier series expansion of this type of fractional trigonometric function.

Lemma 3.1: *If* a,b *are real numbers,* $a>b$, *and* $b\neq 0$, *then*

$$[a+bcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}=\frac{-1}{\sqrt{a^2-b^2}}+\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^{k}cos_{\alpha}(kx^{\alpha}). \tag{16}$$

Proof Let $r=\frac{1}{2}\left(\sqrt{a+b}+\sqrt{a-b}\right)$, $q=\frac{1}{2}\left(\sqrt{a+b}-\sqrt{a-b}\right)$, then

$$r^2+q^2=\frac{1}{4}\cdot 2\cdot[(a+b)+(a-b)]=a, \tag{17}$$

$$2rq=2\cdot\frac{1}{4}\cdot[(a+b)-(a-b)]=b, \tag{18}$$

$$\frac{r}{q}=\frac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}=\frac{2a+2\sqrt{a^2-b^2}}{2b}=\frac{a+\sqrt{a^2-b^2}}{b}, \tag{19}$$

$$\frac{q}{r}=\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}=\frac{2a-2\sqrt{a^2-b^2}}{2b}=\frac{a-\sqrt{a^2-b^2}}{b}. \tag{20}$$

Therefore,

$$[a+bcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cdot\left[\sqrt{a^2-b^2}\,[a+bcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right]$$

$$=\frac{1}{\sqrt{a^2-b^2}}\cdot\left[-1+\left[a+\sqrt{a^2-b^2}+bcos_{\alpha}(x^{\alpha})\right]\otimes_{\alpha}[a+bcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{b}{\sqrt{a^2-b^2}}\left[\frac{a+\sqrt{a^2-b^2}+bcos_{\alpha}(x^{\alpha})}{b}\otimes_{\alpha}[a+bcos_{\alpha}(x^{\alpha})]^{\otimes_{\alpha}(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{b}{\sqrt{a^2-b^2}}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)\right]\otimes_\alpha\ [r^2+q^2+2rqcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{b}{q^2\cdot\sqrt{a^2-b^2}}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)\right]\otimes_\alpha\ \left[\left(\frac{r}{q}\right)^2+2\cdot\frac{r}{q}cos_\alpha(x^\alpha)+1\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{\frac{b}{q^2}}{\sqrt{a^2-b^2}}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)\right]\otimes_\alpha\ \left[\left(\frac{r}{q}+cos_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}+[sin_\alpha(x^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{\frac{b}{q^2}}{\sqrt{a^2-b^2}}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)\right]\otimes_\alpha\ \left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)\right]\otimes_\alpha\left[\frac{r}{q}+cos_\alpha(x^\alpha)-isin_\alpha(x^\alpha)\right]\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{\frac{b}{q^2}}{\sqrt{a^2-b^2}}\mathrm{Re}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)-isin_\alpha(x^\alpha)\right]\otimes_\alpha\ \left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)\right]\otimes_\alpha\left[\frac{r}{q}+cos_\alpha(x^\alpha)-\right.\right.\right.$$

isinaxa⊗α(−1)

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{\frac{b}{q^2}}{\sqrt{a^2-b^2}}\mathrm{Re}\left[\left[\frac{r}{q}+cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{\frac{b}{rq}}{\sqrt{a^2-b^2}}\mathrm{Re}\left[\left[1+\frac{q}{r}E_\alpha(ix^\alpha)\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{2}{\sqrt{a^2-b^2}}\mathrm{Re}\left[\sum_{k=0}^{\infty}\left(-\frac{q}{r}E_\alpha(ix^\alpha)\right)^{\otimes_\alpha k}\right]\ \ (\text{since } \left|-\frac{q}{r}\right|=\left|\frac{q}{r}\right|<1)$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{2}{\sqrt{a^2-b^2}}\mathrm{Re}\left[\sum_{k=0}^{\infty}\left(-\frac{q}{r}\right)^k E_\alpha(ikx^\alpha)\right]\ \ (\text{by fractional DeMoivre's formula})$$

$$=\frac{-1}{\sqrt{a^2-b^2}}+\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^k cos_\alpha(kx^\alpha)\ .\ \ (\text{by fractional Euler's formula})$$

q.e.d.

Theorem 3.2: *Let* $0<\alpha\le 1$, n *be any positive integer,* a, b *be real numbers,* $a>b$, *and* $b\neq 0$. *Then*

$$\left({}_0D_x^\alpha\right)^n\left[[a+bcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]=\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^k k^n\cdot cos_\alpha\left(kx^\alpha+n\cdot\frac{T_\alpha}{4}\right). \tag{21}$$

Proof By Lemma 3.1,

$$\left({}_0D_x^\alpha\right)^n\left[[a+bcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right]$$

$$=\left({}_0D_x^\alpha\right)^n\left[\frac{-1}{\sqrt{a^2-b^2}}+\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^k cos_\alpha(kx^\alpha)\right]$$

$$=\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^k\left({}_0D_x^\alpha\right)^n[cos_\alpha(kx^\alpha)]$$

$$=\frac{2}{\sqrt{a^2-b^2}}\cdot\sum_{k=0}^{\infty}\left(-\frac{a-\sqrt{a^2-b^2}}{b}\right)^k k^n\cdot cos_\alpha\left(kx^\alpha+n\cdot\frac{T_\alpha}{4}\right).$$

q.e.d.

IV. CONCLUSION

In this paper, we use fractional Fourier series to obtain the formula of any order fractional derivative of some type of fractional trigonometric function. Moreover, Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions play important roles in this article. In fact, our result is a generalization of classical calculus result. In the future, we will continue to study the problems in applied mathematics and fractional differential equations by using our methods.

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